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by

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Abstract. We present a characterization of the compact-open topology as a graph topology on the space of continuous functions on a compact Hausdorff space with values in a Tychonoff space.

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1. Introduction

Every continuous function from a topological space to a Hausdorff space has a closed graph in the product space. This identification of a continuous function with its graph enables one to view it as an element of the space of closed subsets of the product space and consequently to endow the space of continuous functions C(X,Y) with the relative topology of set-convergence of Choquet [2] and Kuratowski [7]; see [3, pp. 247-254] for details and references. This point of view is hardly novel and has been used to define a topology on the space of upper semicontinuous functions (see [3] and also [1, Theorem 2.78), and also to consider convergence of sequences of maximal monotone operators on a reflexive Banach space (see [1, pp. 351-352 and 360-361] for details and references). However, a characterization of this topology on the space of continuous functions as the compact-open topology of Fox and Arens (see, for example, [4], [8] or [9]) appears not to have been noticed before. This characterization is the main object of this note.

Our result was motivated by a problem in mathematical economics where the space of continuous functions is viewed as a space of economic agents each of whose preferences depend on the distribution of the actions of the others (see [5] and [6] for details).

2. The Results

For any topological space (X, τ_X), let $\mathfrak{F}(X)$ denote the set of closed subsets of X. Let $\tau^K(X)$ denote the topology of set-convergence, also termed the Kuratowski topology, on $\mathfrak{F}(X)$. $\tau^K(X)$ is generated by a sub-base consisting of

$$\{F \in \mathfrak{F}(X): \ F \cap K = \phi\} \text{ and } \{F \in \mathfrak{F}(X): \ F \cap G \neq \phi\}$$

for all compact sets K in X and open sets G in X.

For any two topological spaces (X, τ_x) and (Y, τ_y), let $C(X,Y)$ denote the set of continuous functions from X to Y. We shall denote the relativization of $\tau^K(X \times Y)$ to $C(X,Y)$ by $\tau^G(X,Y)$ and shall call this relative topology, the <u>graph topology</u> on $C(X,Y)$.

On $C(X,Y)$ the <u>compact-open topology</u> is denoted by $\tau^C(X,Y)$ and is generated by a sub-base consisting of

$$(K,G) = \{f \in C(X,Y): \ f(K) \subset G\}$$

for all compact sets K in X and open sets G in Y.

Before we state our results, we need an elementary lemma.

<u>Lemma 1</u>. <u>Let</u> (X,τ_x), (Y,τ_y), (Z,τ_z) <u>be topological spaces such that</u> (Y,τ_y) <u>is a subspace of</u> (Z, τ_z). <u>Then</u> $C(X,Y) \subset C(X,Z)$.

Lemma 1 allows us to consider the relativization of $\tau^K(X \times Z)$ on $C(X,Y)$. We shall denote this topology by $\tau^G(X,Z)$. We can now state

<u>Theorem</u>. <u>Let</u> (X,τ_x), (Z,τ_z) <u>be compact Hausdorff spaces and</u> (Y,τ_y) <u>a subspace of</u> (Z,τ_z). <u>Then</u> $\tau^C(X,Y)$ <u>is identical to</u> $\tau^G(X,Z)$ <u>on</u> $C(X,Y)$.

This result has a number of corollaries, the first of which is rather straightforward.

<u>Corollary 1</u>. <u>Let</u> (X,τ_x) <u>and</u> (Y,τ_y) <u>be compact Hausdorff spaces. Then</u> $\tau^C(X,Y)$ <u>is identical to</u> $\tau^G(X,Y)$ <u>on</u> $C(X,Y)$.

Our next corollary can be viewed as the principal result of this note. It is phrased in terms of a completely regular T_1-space; also called a Tychonoff space.

Corollary 2. *Let (X,τ_x) be a compact Hausdorff space and (Y,τ_y) be a Tychonoff space. Then for any two compact Hausdorff spaces (Z_1,τ_{z_1}), (Z_2,τ_{z_2}) in which (Y,τ_y) is embedded, the topologies $\tau^G(X,Z_1)$ and $\tau^G(X,Z_2)$ are identical on $C(X,Y)$ and equal to $\tau^C(X,Y)$ on $C(X,Y)$.*

We can now use the characterization of the compact-open topology given in Corollary 2 to deduce the following.

Corollary 3. *Let (X,τ_x) be a compact Hausdorff space and (Y,τ_y) be a Tychonoff space. Then $(C(X,Y), \tau^C(X,Y)$ is a Tychonoff space.*

Corollary 3 is, of course, a well-known result even for the case when (X,τ) is not necessarily compact Hausdorff. However, our proof for the compact case is very different from that presented in Nagata [8, pp. 272-274] or Willard [9, p. 288, Exercise 43B with Theorem 43.7 as a hint].

3. Proofs

We begin with a

Proof of Lemma 1.

Pick any $f \in C(X,Y)$ and $G \in \tau_z$. Certainly $G = (G \cap Y) \cup (G-Y)$. Then $f^{-1}(G) = f^{-1}(G \cap Y) \cup f^{-1}(G-Y)$. Since $(G \cap Y) \in \tau_y$ and $f^{-1}(G-Y) = \phi$, we are done.

$\square$

Before we present a proof of the theorem, we record two elementary results for which we could find no reference.

Lemma 2. Let $B(\Sigma)$ denote all finite intersections of elements of Σ. Then a sufficient condition that two sub-bases Σ, Σ' in X lead to an identical topology on X is that

(i) For each $U \in \Sigma$ and each $x \in U$, there is a $U' \in B(\Sigma')$ with $x \in U' \subset U$;

(ii) for each $U' \in \Sigma'$ and each $x \in U'$, there is a $U \in B(\Sigma)$ with $x \in U \subset U'$.

Proof. The proof is a simple consequence of [4, III.3.4].

$\square$

Lemma 3. Let (X,τ_x) be a compact Hausdorff space and $G_i \in \tau_x$, $G_i \neq \phi$, $i=1, \ldots, n$, such that $X = \bigcup_{i=1}^{n} G_i$. Then there exist compact sets $K_i \subset G_i$, such that $X = \bigcup_{i=1}^{n} K_i$.

Proof. The assertion is trivially true for $n=1$. Suppose it is true for $n-1$. Let $L = \bigcup_{i=1}^{n-1} G_i$. Since $L \cup G_n = X$, $L^c \cap G_n^c = \phi$. Since (X,τ_x) is compact Hausdorff, it is normal [4, XI.1.2], and hence there exist 0_1 and 0_2 in τ_x such that $0_1 \cap 0_2 = \phi$ and $L^c \subset 0_1$ and $G_n^c \subset 0_2$. Certainly 0_1^c and 0_2^c are compact and nonempty and such that $0_1^c \subset L$, $0_2^c \subset G_n$ and $0_1^c \cup 0_2^c = X$.

Now endow 0_1^c with the relativization of τ_x and denote it by τ_x^0. Since 0_1^c is closed in (X,τ_x), $(0_1^c, \tau_x^0)$ is a compact Hausdorff space [4, XI.1.4(3)]. Furthermore $(0_1^c \cap G_i) \in \tau_x^0$. Since

$$O_1^c = \bigcup_{i=1}^{n-1} (O_1^c \cap G_i),$$

the induction hypothesis applies and we can find compact sets L_i in (O^c, τ_x) such that $O_1^c = \bigcup_{i=1}^{n-1} L_i$. Since L_i are closed in (X,τ_x), [4, III.7.3], $X = O_2^c \cup (\bigcup_{i=1}^{n} L_i)$ and we are done.

$\square$

Proof of Theorem. The first point to be noted is that, following [4, III.7.2(1)], the sub-base for $\tau^G(XxZ)$ consists of

$$[K] = \{f \in C(X,Y): \text{ graph } f \cap K = \phi\}$$

$$\langle G\rangle = \{f \in C(X,Y): \text{ graph } f \cap G \neq \phi\}$$

for all compact sets K in X x Z and open sets G in X x Z. We shall now use Lemma 2 to prove the theorem.

We first show (i) and pick (K,G) and $f \in (K,G)$. Since $G \in \tau_y$, there exists $H \in \tau_z$ such that $G = H \cap Y$. Since $f(K) \subset G$ and $f(K) \subset Y$, $f(K) \subset H$. Hence, graph $f \cap (K \times H^c) = \phi$ and since (Z, τ_z) is compact Hausdorff, $f \in [K \times H^c]$. This also shows that $[K \times H^c] \subset [K,G]$ and we are done.

Next, pick [K] and $f \in [K]$. Since K^c is open in X x Z, there exists an index set I and open sets X_i and G_i in X and Z respectively such that $K^c = \bigcup_{i\in I}(X_i \times G_i)$. Since $f \in [K]$, graph f is contained in K^c and hence in $\bigcup_{i\in I}(X_i \times G_i)$. Since graph f is a compact set in X x Z, there exists a finite integer n such that

$$\text{graph } f \subset \bigcup_{i=1}^{n} (X_i \times G_i).$$

But this can be rewritten as

$$\text{graph } f = \bigcup_{i=1}^{n} (\text{graph } f) \cap (X_i \times G_i).$$

But (graph $f \cap (X_i \times G_i)$) is an open set in graph f and hence we can appeal to Lemma 3 to find sets L_i, compact in graph f, such that graph $f = \bigcup_{i=1}^{n} L_i$ and

$$(*) \qquad L_i \subset (\text{graph } f \cap (X_i \times G_i)) \qquad i = 1, \ldots, n.$$

Without loss of generality, let $L_i \neq \phi$ for all $i=1, \ldots, n$. Now let $K_i = \text{proj}_X L_i$. Since L_i are closed in graph f and since graph f is closed in $(X \times Z, \tau_{x \times z})$, and since the projection map is a closed map, K_i is compact in (X, τ_x). Furthermore, $\bigcup_{i=1}^{n} K_i = X$ and from (*) $K_i \subset X_i$ for all $i=1, \ldots, n$. For any $x \in K_i$, $(x,f(x)) \in L_i \subset (X_i \times G_i)$. Hence $f(K_i) \subset G_i$. Let $H_i = G_i \cap Y$ and observe that $f \in \bigcap_{i=1}^{n} (K_i, H_i)$. Since $H_i \in \tau_y$ and K_i are compact in (X,τ_x), (K_i,H_i) is a sub-basic set for $\tau^c(X,Y)$. Now pick any $g \in \bigcap_{i=1}^{n} (K_i,H_i)$. For all $i=1, \ldots, n$, $g(K_i) \subset H_i$ and hence graph $g \subset \bigcup_{i=1}^{n} (K_i \times H_i) \subset \bigcup_{i=1}^{n} (X_i \times G_i) \subset K^c$ and hence graph $g \cap K = \phi$. This implies $g \in [K]$. Since g was arbitrary,

$$\bigcap_{i=1}^{n} (K_i,H_i) \subset [K]$$

and we are done.

For our final step, pick $\langle G\rangle$ and $f \in \langle G\rangle$. Since graph $f \cap G \neq \phi$, there exists $x \in X$ such that $(x,f(x)) \in G$. Since $G \in \tau_{xxz}$, there

exists $X_1 \in \tau_x$ and $H \in \tau_z$ such that $(X_1 \times H) \subset G$ and $(x,f(x)) \in X_1 \times H$. Let $H' = H \cap Y$. Then $f \in (\{x\}, H')$. Since $\{x\}$ is compact in (X,τ_x) and $H' \in \tau_y$, $(\{x\},H')$ is a sub-basic set for $\tau^c(X,Y)$.

Now pick any $g \in (\{x\},H')$. Since $g(x) \subset H' = (H \cap Y)$, $(x,g(x)) \in (X_1 x H) \subset G$. Hence graph $g \cap G \neq \phi$ and $g \in \langle G\rangle$. Since g was arbitrary, we have shown

$$[\{x\}, H'] \subset \langle G\rangle.$$

We can now appeal to Lemma 1 to finish the proof of the theorem.

□

Proof of Corollary 1. This is obvious once we choose Y to be equal to Z.

□

Proof of Corollary 2. The fact that this embedding is always possible is guaranteed by [9, Theorem 14.13]. The result now follows directly from the theorem.

□

Proof of Corollary 3. From Corollary 2, we know that $\tau^c(X,Y)$ is identical to $\tau^G(X,Z)$, where (Z,τ_z) is any compact Hausdorff space in which (Y,τ_y) is embedded. Now $\tau^G(X,Z)$ is the relativization of $\tau^K(XxZ)$. Since XxZ is a compact Hausdorff space, $(XxZ, \tau^K(XxZ))$, is a compact Hausdorff space [1, Theorem 2.76]. Hence $\tau^K(XxZ)$ is normal and therefore Tychonoff. Since every subspace of a Tychonoff space is Tychonoff [9, Theorem 14.10(a)], we are done.

□

References

[1] H. Attouch, Variational Convergence for Functions and Operators, Pitman Advanced Publishing Program, London, 1984.

[2] G. Choquet, "Convergences," Ann. Univ. de Grenoble, 23 (1947-48) 55-112.

[3] S. Dolecki, G. Salinetti and R. Wets, "Convergence of Functions; Equi-Semicontinuity," Trans. Amer. Math. Soc., 276 (1983) 409-429.

[4] J. Dugundji, Topology, Allyn and Bacon Inc., Boston, 1966.

[5] M. Ali Khan, "On A Variant of a Theorem of Schmeidler," BEBR Faculty Paper No. 1306, University of Illinois. Revised in December 1986.

[6] M. Ali Khan and Y. Sun, "On A Reformulation of Cournot-Nash Equilibrium," mimeo.

[7] K. Kuratowski, Topology, Academic Press, New York, 1966.

[8] J. Nagata, Modern General Topology, North Holland Publishing Co., Amsterdam, 1968.

[9] S. Willard, General Topology, Addison-Wesley, New York, 1970.